ALGEBRAIC APPROACH TO QUANTUM FIELD THEORY

ROMEO BRUNETTI AND KLAUS FREDENHAGEN
II INST. F. THEOR. PHYSIK, UNIV. HAMBURG
149 LURUPER CHAUSSEE, D-22761 HAMBURG, GERMANY

1. Introduction

Quantum Field Theory may be understood as the incorporation of the principle of locality, which is at the basis of classical field theory, into quantum physics. There are, however, severe obstacles against a straightforward translation of concepts of classical field theory into quantum theory, among them the notorious divergences of quantum field theory and the intrinsic non-locality of quantum physics. Therefore, the concept of locality is somewhat obscured in the formalism of quantum field theory as it is typically exposed in textbooks. Nonlocal concepts as the vacuum, the notion of particles or the S-matrix play a fundamental rôle, and neither the relation to classical field theory nor the influence of background fields can be properly treated.

Algebraic quantum field theory (synonymously, Local Quantum Physics), on the contrary, aims at emphasizing the concept of locality at every instance. As the nonlocal features of quantum physics occur at the level of states ("entanglement"), not at the level of observables, it is better not to base the theory on the Hilbert space of states but on the algebra of observables. Subsystems of a given system then simply correspond to subalgebras of a given algebra. The locality concept is abstractly encoded in a notion of independence of subsystems; two subsystems are independent if the algebra of observables which they generate is isomorphic to the tensor product of the algebras of the subsystems.

Spacetime can then – in the spirit of Leibniz – be considered as an ordering device for systems. So one associates to regions of spacetime the algebras of observables which can be measured in the pertinent region, with the condition that the algebras of subregions of a given region can be identified with subalgebras of the algebra of the region.

Problems arise if one aims at a generally covariant approach in the spirit of General Relativity. Then, in order to avoid pitfalls like in the "hole problem", systems corresponding to isometric regions must be isomorphic. Since isomorphic regions may be embedded into different spacetimes, this amounts to a simultaneous treatment of all spacetimes of a suitable class. We shall see that category theory furnishes such a description, where the objects are the systems and the morphisms the embeddings of a system as a subsystem of other systems.

States arise as secondary objects via Hilbert space representations, or directly as linear functionals on the algebras of observables which can be interpreted as expectation values and are therefore positive and normalized.

It is crucial that inequivalent representations ("sectors") can occur, and the analysis of the structure of the sectors is one of the big successes of Algebraic Quantum Field Theory. One also can study the particle interpretation of certain states as well as (equilibrium and non-equilibrium) thermodynamical properties.

The mathematical methods in Algebraic Quantum Field Theory are mainly taken from the theory of operator algebras, a field of mathematics which developped in close contact to mathematical physics, in particular to Algebraic Quantum Field Theory. Unfortunately, the most important field theories, from the point of view of elementary particle physics, as Quantum Electro Dynamics or the Standard Model could not yet be constructed beyond formal perturbation theory with the annoying consequence that it seemed that the concepts of Algebraic Quantum Field Theory could not be applied to them. But recently, it was shown that formal perturbation theory can be reshaped in the spirit of Algebraic Quantum Field Theory such that the algebras of observables of these models can be constructed as algebras of formal power series of Hilbert space operators. The price to pay is that the deep mathematics of operator algebras cannot be applied, but the crucial features of the algebraic approach can be used.

Algebraic Quantum Field Theory was originally proposed by Haag as a concept by which scattering of particles can be understood as a consequence of the principle of locality. It was then put into a mathematically precise form by Araki, Haag and Kastler. After the analysis of particle scattering by Haag and Ruelle and the clarification of the relation to the Lehmann-Symanzik-Zimmermann formalism by Hepp, the structure of superselection sectors was studied first by Borchers and then in a fundamental series of papers by Doplicher, Haag and Roberts, soon after Buchholz and Fredenhagen established the relation to particles, and finally Doplicher and Roberts uncovered the structure of superselection sectors as the dual of a compact group thereby generalizing the Tannaka-Krein Theorem of characterization of group duals.

With the advent of two-dimensional Conformal Field Theory new models were constructed and it was shown that the DHR analysis can be generalized to these models. Directly related to conformal theories is the algebraic approach to holography in anti-de Sitter (AdS) spacetime by Rehren.

The general framework of Algebraic Quantum Field Theory may be described as a covariant functor between two categories. The first one contains the information on local relations and is crucial for the interpretation. Its objects are topological spaces with additional structures (typically globally hyperbolic Lorentzian spaces, possibly spin bundles with connections, etc.), its morphisms structure preserving embeddings. In the case of globally hyperbolic Lorentzian spacetimes one requires that the embeddings are isometric and preserve the causal structure. The second category describes the algebraic structure of observables, in quantum physics the standard assumption is that one deals with the category of C^*-algebras where the morphisms are unital embeddings. In classical physics one looks instead at Poisson algebras, and in perturbative quantum field theory one admits algebras which

possess nontrivial representations as formal power series of Hilbert space operators. It is the leading principle of Algebraic Quantum Field Theory that the functor $\mathscr{A}$ contains all physical information. In particular, two theories are equivalent if the corresponding functors are naturally equivalent.

In the analysis of the functor $\mathscr{A}$ a crucial rôle is played by natural transformations from other functors on the locality category. For instance, a field A may be defined as a natural transformation from the category of test function spaces to the category of observable algebras via their functors related to the locality category.

2. Quantum Field Theories as Covariant Functors

The rigorous implementation of the generally covariant locality principle uses the language of category theory.

The following two categories are used :

Loc: The class of objects obj(**Loc**) is formed by all (smooth) d-dimensional ($d \geq 2$ is held fixed), globally hyperbolic Lorentzian spacetimes M which are oriented and time-oriented. Given any two such objects M_1 and M_2, the morphisms $\psi \in \mathrm{hom}_{\textbf{Loc}}(M_1, M_2)$ are taken to be the isometric embeddings $\psi : M_1 \to M_2$ of M_1 into M_2 but with the following constraints;

(i) if $\gamma : [a, b] \to M_2$ is any causal curve and $\gamma(a), \gamma(b) \in \psi(M_1)$ then the whole curve must be in the image $\psi(M_1)$, i.e., $\gamma(t) \in \psi(M_1)$ for all $t \in]a, b[$;

(ii) any morphism preserves orientation and time-orientation of the embedded spacetime.

Composition is composition of maps, the unit element in $\mathrm{hom}_{\textbf{Loc}}(M, M)$ is given by the identical embedding $\mathrm{id}_M : M \mapsto M$ for any $M \in$ obj(**Loc**).

Obs: The class of objects obj(**Obs**) is formed by all C*-algebras possessing unit elements, and the morphisms are faithful (injective) unit-preserving $*$-homomorphisms. The composition is again defined as the composition of maps, the unit element in $\mathrm{hom}_{\textbf{Obs}}(\mathcal{A}, \mathcal{A})$ is for any $\mathcal{A} \in$ obj(**Obs**) given by the identical map $\mathrm{id}_\mathcal{A} : A \mapsto A$, $A \in \mathcal{A}$.

The choice of the categories is done for definitiveness. One may envisage changes according to particular needs, as for instance in perturbation theory where instead of C*-algebras general topological $*$-algebras are better suited. Or one may use von Neumann algebras, in case particular states are selected. On the other side, one might consider for **Loc** bundles over spacetimes, or one might (in conformally invariant theories) admit conformal embeddings as morphisms. In case one is interested in spacetimes which are not globally hyperbolic one could look at the globally hyperbolic subregions (where care has to be payed to the causal convexity condition (i) above).

Now we define the concept of locally covariant quantum field theory.

Definition 2.1.

(i) *A* **locally covariant quantum field theory** *is a covariant functor $\mathscr{A}$*

from **Loc** *to* **Obs** *and (writing α_ψ for $\mathscr{A}(\psi)$) with the covariance properties*

$$\alpha_{\psi'} \circ \alpha_\psi = \alpha_{\psi' \circ \psi}\,, \quad \alpha_{\mathrm{id}_M} = \mathrm{id}_{\mathscr{A}(M)}\,,$$

for all morphisms $\psi \in \hom_{\mathbf{Loc}}(M_1, M_2)$, all morphisms $\psi' \in \hom_{\mathbf{Loc}}(M_2, M_3)$ and all $M \in \mathrm{obj}(\mathbf{Loc})$.

(ii)   A locally covariant quantum field theory described by a covariant functor $\mathscr{A}$ is called **causal** *if the following holds: Whenever there are morphisms $\psi_j \in \hom_{\mathbf{Loc}}(M_j, M)$, $j = 1, 2$, so that the sets $\psi_1(M_1)$ and $\psi_2(M_2)$ are causally separated in M, then one has*

$$[\alpha_{\psi_1}(\mathscr{A}(M_1)), \alpha_{\psi_2}(\mathscr{A}(M_2))] = \{0\}\,,$$

where the element-wise commutation makes sense in $\mathscr{A}(M)$.

(iii)   We say that a locally covariant quantum field theory given by the functor $\mathscr{A}$ obeys the **time-slice axiom** *if*

$$\alpha_\psi(\mathscr{A}(M)) = \mathscr{A}(M')$$

holds for all $\psi \in \hom_{\mathbf{Loc}}(M, M')$ such that $\psi(M)$ contains a Cauchy-surface for M'.

Thus, a quantum field theory is an assignment of C*-algebras to (all) globally hyperbolic spacetimes so that the algebras are identifiable when the spacetimes are isometric, in the indicated way. This is a precise description of the **generally covariant locality principle**.

3. THE TRADITIONAL APPROACH

The traditional framework of algebraic quantum field theory, in the Araki-Haag-Kastler sense, on a fixed globally hyperbolic spacetime can be recovered from a locally covariant quantum field theory, i.e. from a covariant functor $\mathscr{A}$ with the properties listed above.

Indeed, let M be an object in $\mathrm{obj}(\mathbf{Loc})$. We denote by $\mathcal{K}(M)$ the set of all open subsets in M which are relatively compact and contain with each pair of points x and y also all $\boldsymbol{g}$-causal curves in M connecting x and y (cf. condition (i) in the definition of **Loc**). $O \in \mathcal{K}(M)$, endowed with the metric of M restricted to O and with the induced orientation and time-orientation is a member of $\mathrm{obj}(\mathbf{Loc})$, and the injection map $\iota_{M,O} : O \to M$, i.e. the identical map restricted to O, is an element in $\hom_{\mathbf{Loc}}(O, M)$. With this notation it is easy to prove the following assertion:

Theorem 3.1. *Let $\mathscr{A}$ be a covariant functor with the above stated properties, and define a map $\mathcal{K}(M) \ni O \mapsto \mathcal{A}(O) \subset \mathscr{A}(M)$ by setting*

$$\mathcal{A}(O) := \alpha_{\iota_{M,O}}(\mathscr{A}(O))\,.$$

Then the following statements hold:

(a) *The map fulfills isotony, i.e.*

$$O_1 \subset O_2 \Rightarrow \mathcal{A}(O_1) \subset \mathcal{A}(O_2) \qquad \textit{for all} \qquad O_1, O_2 \in \mathcal{K}(M)\,.$$

(b) *The group G of isometric diffeomorphisms $\kappa : M \rightarrow M$ (so that $\kappa_* \boldsymbol{g} = \boldsymbol{g}$) preserving orientation and time-orientation, is represented by C^*-algebra automorphisms $\alpha_\kappa : \mathscr{A}(M) \rightarrow \mathscr{A}(M)$ such that*

$$\alpha_\kappa(\mathcal{A}(O)) = \mathcal{A}(\kappa(O)) , \quad O \in \mathcal{K}(M) .$$

(c) *If the theory given by $\mathscr{A}$ is additionally causal, then it holds that*

$$[\mathcal{A}(O_1), \mathcal{A}(O_2)] = \{0\}$$

for all $O_1, O_2 \in \mathcal{K}(M)$ with O_1 causally separated from O_2.

These properties are just the basic assumptions of the Araki-Haag-Kastler framework.

4. THE ACHIEVEMENTS OF THE TRADITIONAL APPROACH

In the Araki-Haag-Kastler approach in Minkowski spacetime $\mathbb{M}$ a great deal of results have been disclosed in the last forty years, some of those resulting in source of inspiration also to mathematics. Let us organize the description of the achievements in terms of a length-scale basis, from the small to the large. We assume in this section that the algebra $\mathscr{A}(\mathbb{M})$ is faithfully and irreducibly represented on a Hilbert space $\mathcal{H}$, that the Poincaré transformations are unitarily implemented with positive energy, and that the subspace of Poincaré invariant vectors is one dimensional (uniqueness of the vacuum). Moreover, algebras correponding to regions which are spacelike to a nonempty open region are assumed to be weakly closed (i.e. von Neumann algebras on $\mathcal{H}$), and the condition of weak additivity is fulfilled, i.e. for all $O \in \mathcal{K}(\mathbb{M})$ the algebra generated from the algebras $\mathcal{A}(O + x)$, $x \in \mathbb{M}$ is weakly dense in $\mathscr{A}(\mathbb{M})$.

4.1. Ultraviolet structure and idealized localizations. This part deals with the problem of inspecting the theory at very small scales. At the extreme we are interested in idealized localizations, eventually to points of spacetimes. But the observable algebras are trivial at any point $x \in \mathbb{M}$, namely

$$\bigcap_{O \ni x} \mathcal{A}(O) = \mathbb{C}\mathbf{1} , \qquad O \in \mathcal{K}(\mathbb{M}) .$$

Hence pointlike localized observables are necessarily singular. Actually, the Wightman formulation of quantum field theory is based on the use of distributions on spacetime with values in the algebra of observables (as a topological *-algebra). In spite of technical complications whose physical significance is unclear this formalism is well suited for a discussion of the connection with the euclidean theory which allows in fortunate cases a treatment by path integrals; it is more directly related to models and admits via the operator product expansion a study of the short distance behaviour. It is therefore an important question how the algebraic approach is related to the Wightman formalism. We refer to the literature for exploring the results on this relation.

Whereas these results point to an essential equivalence of both formalisms, one needs in addition a criterion for the existence of sufficiently many Wightman fields associated to a given local net. Such a criterion can be given in terms of a compactness condition to be discussed in the next subsection.

As a benefit one derives an operator product expansion which has to be assumed in the Wightman approach.

In the purely algebraic approach the ultraviolet structure has been investigated by Buchholz and Verch. Small scale properties of theories are studied with the help of the so called scaling algebras whose elements can be described as orbits of observables under all possible renormalization group motions. There results a classification of theories in the scaling limit which can be synthetised into three broad classes; theories for which the scaling limit is purely classical (commutative algebras), those for which the limit is essentially unique (stable ultraviolet fixed point) and not classical and those for which this is not the case (unstable ultraviolet fixed point). This classification does not rely on perturbation expansions. It allows an intrinsic definition of confinement in terms of so-called ultraparticles, i.e. particles which are visible only in the scaling limit.

4.2. **Phase space analysis.** As far as finite distances are concerned there are two apparently competing principles, those of nuclearity and modularity. The first one suggests that locally, after a cutoff in energy, one has a situation similar to that of old quantum mechanics, namely, a finite number of states in a finite volume of phase space. Aiming at a precise formulation Haag and Swieca introduced their notion of compactness, which Buchholz and Wichmann sharpened into that of nuclearity. The latter authors proposed that the set generated from the vacuum vector Ω

$$\{e^{-\beta H} A\Omega \mid A \in \mathcal{A}(O) \ , \ \|A\| < 1\} \, ,$$

H denoting the generator of time translations (Hamiltonian), is nuclear for any $\beta > 0$, roughly saying that it is contained in the image of the unit ball under a trace class operator. The nuclear size $Z(\beta, O)$ of the set plays the rôle of the partition function of the model and has to satisfy certain bounds in the parameter β. The consequence of this constraint is the existence of product states, namely those normal states for which observables localized in two given space-like separated regions are uncorrelated. A further consequence is the existence of thermal equilibrium states (KMS states) for all $\beta > 0$.

The second principle concerns the fact that even locally, quantum field theory has infinitely many degrees of freedom. This becomes visible in the Reeh-Schlieder Theorem which states that every vector Φ which is in the range of $e^{-\beta H}$ for some $\beta > 0$ (in particular the vacuum Ω) is cyclic and separating for the algebras $\mathcal{A}(O)$, $O \in \mathcal{K}(\mathbb{M})$, i.e. $\mathcal{A}(O)\Phi$ is dense in $\mathcal{H}$ (Φ is cyclic) and $A\Phi = 0$, $A \in \mathcal{A}(O)$ implies $A = 0$ (Φ is separating). The pair $(\mathcal{A}(O), \Omega)$ is then a von Neumann algebra in the so called *standard form*. On such a pair the Tomita-Takesaki theory can be applied, namely the densely defined operator

$$SA\Omega = A^*\Omega \, , \, A \in \mathcal{A}(O) \, ,$$

is closable, and the polar decomposition of its closure $\bar{S} = J\Delta^{1/2}$ delivers an antiunitary involution J (the modular conjugation) and a positive self-adjoint operator Δ (the modular operator) associated to the standard pair $(\mathcal{A}(O), \Omega)$. These operators have the properties

$$J\mathcal{A}(O)J = \mathcal{A}(O)'$$

where the prime denotes the commutant, and

$$\Delta^{it}\mathcal{A}(O)\Delta^{-it} = \mathcal{A}(O), \ t \in \mathbb{R}.$$

The importance of this structure is based on the fact disclosed by Bisognano and Wichmann using Poincaré covariant Wightman fields and local algebras generated by them, that for specific regions in Minkowski spacetime the modular operators have a geometrical meaning. Indeed, these authors showed for the pair $(\mathcal{A}(W),\Omega)$ where W denotes the wedge region $W = \{x \in \mathbb{M} \mid |x^0| < x^1\}$ that the associated modular unitary Δ^{it} is the Lorentz boost with velocity $\tanh(2\pi t)$ in the direction 1 and that the modular conjugation J is the CP_1T symmetry operator with parity P_1 the reflection w.r.t. the $x^1 = 0$ plane. Later, Borchers discovered that already on the purely algebraic level a corresponding structure exists. He proved that given any standard pair $(\mathcal{A}, \Phi)$ and a one-parameter group of unitaries $\tau \to U(\tau)$ acting on the Hilbert space $\mathcal{H}$ with a positive generator and such that Φ is invariant and $U(\tau)\mathcal{A}U(\tau)^* \subset \mathcal{A}$, $\tau > 0$, then the associated modular operators Δ and J fulfil the commutation relations

$$\begin{aligned}
\Delta^{it}U(\tau)\Delta^{-it} &= U(e^{-2\pi t}\tau) \\
JU(\tau)J &= U(-\tau)
\end{aligned}$$

which are just the commutation relations between boosts and light-like translations.

Surprisingly, there is a direct connection between the two concepts of nuclearity and modularity. Indeed, in the nuclearity condition it is possible to replace the Hamiltonian operator by a specific function of the modular operator associated to a slightly larger region. Furthermore under mild conditions nuclearity and modularity together determine the structure of local algebras completely; they are isomorphic to the unique hyperfinite type III_1 von Neumann algebra.

4.3. **Sectors, symmetries, statistics and particles.** Large scales are appropriate for discussing global issues like superselection sectors, statistics and symmetries as far as large spacelike distances are concerned and scattering theory, with the resulting notions of particles and infraparticles, as far as large timelike distances are concerned.

In purely massive theories where the vacuum sector has a mass gap and where the mass shell of the particles are isolated, a very satisfactory description of the multi-particle structure at large times can be given. Using the concept of almost local particle generators,

$$\Psi = A(t)\Omega$$

where Ψ is a single particle state (i.e. an eigenstate of the mass operator), $A(t)$ is a family of almost local operators essentially localized in the kinematical region accessible from a given point by a motion with the velocities contained in the spectrum of Ψ, one obtains the multi-particle states as limits of products $A_1(t) \cdots A_n(t)\Omega$ for disjoint velocity supports. The corresponding closed subspaces are invariant under Poincaré transformations and are unitarily equivalent to the Fock spaces of non-interacting particles.

For massless particles, no almost local particle generators can be expected to exist. In even dimensions, however, one can exploit Huygens principle to construct asymptotic particle generators which are in the commutant of the algebra of the forward or backward lightcone, respectively. Again, their products can be determined and deliver multiparticle states.

Much less well understood is the case of massive particles in a theory which possesses also massless particles. Here, in general, the corresponding states are not eigenstates of the mass operator. Since QED as well as the standard model of elementary particles have this problem the correct treatment of scattering in these models is still under discussion. One attempt to a correct treatment is based on the concept of so-called particle weights, i.e. unbounded positive functionals on a suitable algebra. This algebra is generated by positive almost local operators annihilating the vacuum and interpreted as counters.

The structure at large spacelike scales may be analyzed by the theory of superselection sectors. The best understood case is that of locally generated sectors which are the objects of the DHR theory. Starting from a distinguished representation π_0(vacuum representation) which is assumed to fulfil Haag duality,

$$\pi_0(\mathcal{A}(O)) = \pi_0(\mathcal{A}(O'))'$$

for all double cones O, one may look at all representations which are equivalent to the vacuum representation if restricted to the observables localized in double cones in the spacelike complement of a given double cone. Such representations give rise to endomorphisms of the algebra of observables, and the product of endomorphisms can be interpreted as a product of sectors ("fusion"). In general, these representations violate Haag duality, but there is a subclass of so-called finite statistics sectors where the violation of Haag duality is small, in the sense that the nontrivial inclusion

$$\pi(\mathcal{A}(O)) \subset \pi(\mathcal{A}(O'))' \ ,$$

has a finite Jones index. These sectors form (in at least 3 spacetime dimensions) a symmetric tensor category with some further properties which can be identified, in a generalization of the Tannaka-Krein Theorem, as the dual of a unique compact group. This group plays the rôle of a global gauge group. The symmetry of the category is expressed in terms of a representation of the symmetric group. One may then enlarge the algebra of observables and obtains an algebra of operators which transform covariantly under the global gauge group and satisfy Bose- or Fermi commutation relations for spacelike separation.

In 2 spacetime dimensions one obtains instead braided tensor categories. They have been classified under additional conditions (conformal symmetry, complete rationality, $c < 1$). To some extent they can be interpreted as duals of generalized quantum groups.

Concerning the question, whether all representations describing elementary particles are, in the massive case, DHR representations, one can show that in the case of a representation with an isolated mass shell there is an associated vacuum representation which becomes equivalent to the particle representation after restriction to observables localized spacelike to a given

infinitely extended spacelike cone. This property is weaker than the DHR condition but allows in 4 spacetime dimensions the same construction of a global gauge group and of covariant fields with Bose- and Fermi commutation relations, respectively, as the DHR condition. In 3 space dimensions, however, one finds a braided tensor category, which has similar properties as those known from topological field theories in 3 dimensions.

The sector structure in massless theories is, due to the infrared problem, not well understood. This is in particular true for QED.

5. Fields as Natural Transformations

In order to be able to interpret the theory in terms of measurements one has to be able to compare observables associated to different regions of spacetime, or, even different spacetimes. In the absence of nontrivial isometries such a comparison can be made in terms of locally covariant fields. By definition these are natural transformations from the functor of quantum field theory to another functor on the category of spacetimes **Loc**.

The standard case is the functor which associates to every spacetime M its space $\mathcal{D}(M)$ of smooth compactly supported test functions. There the morphisms are the pushforwards $\mathcal{D}\psi \equiv \psi_*$.

Definition 5.1.
*A **locally covariant quantum field** Φ is a natural transformation between the functors $\mathscr{D}$ and $\mathscr{A}$, i.e., for any object M in **Loc** there exists a morphism $\Phi_M : \mathcal{D}(M) \to \mathscr{A}(M)$ such that for any pair of objects M_1 and M_2 and any morphism ψ between them, the following diagram*

$$
\begin{array}{ccc}
\mathcal{D}(M_1) & \xrightarrow{\Phi_{M_1}} & \mathcal{A}(M_1) \\
\psi_* \downarrow & & \downarrow \alpha_\psi \\
\mathcal{D}(M_2) & \xrightarrow[\Phi_{M_2}]{} & \mathcal{A}(M_2)
\end{array}
$$

commutes.

The commutativity of the diagram means, explicitly, that

$$\alpha_\psi \circ \Phi_{M_1} = \Phi_{M_2} \circ \psi_*$$

which is the seeked requirement for the covariance of fields. It contains, in particular the standard covariance condition for spacetime isometries.

Fields in the sense above are not necessarily linear. Examples for fields which are also linear are the scalar massive free Klein Gordon field on all globally hyperbolic spacetimes and its locally covariant Wick polynomials. In particular, the energy momentum tensors can be constructed as locally covariant fields and they provide a crucial tool for discussing the back reaction problem for matter fields.

An example for the more general notion of a field are the local S-matrices in the Stückelberg-Bogolubov-Epstein-Glaser sense. These are unitaries $S_M(\lambda)$ with $M \in$ **Loc** and $\lambda \in \mathcal{D}(M)$ which satisfy the conditions

$$S_M(0) = 1$$

$$S_M(\lambda + \mu + \nu) = S_M(\lambda + \mu)S_M(\mu)^{-1}S_M(\mu + \nu)$$

for $\lambda, \mu, \nu \in \mathcal{D}(M)$ such that the supports of λ and ν can be separated by a Cauchy surface of M with suppλ in the future of the surface.

The importance of these S-matrices relies on the fact that they can be used to define a new quantum field theory. The new theory is locally covariant if the original theory was and if the local S-matrices satisfy the condition of a locally covariant field above. A perturbative construction of interacting quantum field theory on globally hyperbolic spacetimes was completed in this way by Hollands and Wald, based on previous work by Brunetti and Fredenhagen.

References

[1] Araki H (1999) *Mathematical theory of quantum fields*. Oxford: Oxford University Press.

[2] Baumgärtel H and Wollemberg M (1992) *Causal nets of operator algebras*. Berlin: Akademie Verlag.

[3] Borchers HJ (1996) *Translation group and particle representation in quantum field theory*. Lecture Notes in Physics. New Series m: Monographs, 40. Berlin: Springer–Verlag.

[4] Borchers HJ (2000) On revolutionizing quantum field theory with Tomita's modular theory. *Journal of mathematical Physics* 41: 3604-3673.

[5] Bratteli O and Robinson D.W. (1987) *Operator algebras and quantum statistical mechanics, vol. 1*. Berlin: Springer-Verlag.

[6] Brunetti R and Fredenhagen K (2000) Microlocal analysis and interacting quantum field theories: Renormalization on physical backgrounds. *Communications in Mathematical Physics* 208: 623-661.

[7] Brunetti R, Fredenhagen K and Verch R (2003) The generally covariant locality principle – A new paradigm for local quantum field theory. *Communications in Mathematical Physics* 237: 31-68.

[8] Buchholz D and Haag R (2000) The quest for understanding in relativistic quantum physics. *Journal of Mathematical Physics* 41: 3674-3697.

[9] Dixmier J. (1964) *Les C^*-algèbres et leurs représentations*. Paris: Gauthier-Villars.

[10] Doplicher S, Haag R and Roberts JE (1971) Local observables and particle statistics I. *Communications in Mathematical Physics* 23: 199-230.

[11] Doplicher S, Haag R and Roberts JE (1974) Local observables and particle statistics II. *Communications in Mathematical Physics* 35: 49-85.

[12] Evans DE and Kawahigashi Y (1998) *Quantum symmetries on operator algebras*. New York: Clarendon Press, Oxford University Press.

[13] Haag R (1996) *Local Quantum Physics*. Berlin, Heidelberg, New York: Springer-Verlag, 2nd ed.

[14] Haag R and Kastler D (1964) An algebraic approach to quantum field theory. *Journal of Mathematical Physics* 5: 848-861.

[15] Hollands S and Wald RM (2001) Local Wick polynomials and time ordered products of quantum fields in curved spacetime. *Communications in Mathematical Physics* 223: 289-326.

[16] Hollands S and Wald RM (2002) Existence of local covariant time ordered products of quantum field in curved spacetime. *Communications in Mathematical Physics* 231: 309-345.

[17] Kastler D (ed.) (1990) *The algebraic theory of superselection sectors. Introductions and recent results*. Singapore, New Jersey, London, Hong Kong: World Scientific.

[18] Takesaki M (2003) *Theory of operator algebras I, II, III*. Encyclopedia of Marthematical Sciences, 124, 125, 127. Berlin: Springer–Verlag.

[19] Wald RM (1994) *Quantum field theory in curved spacetime and black hole thermodynamics*. Chicago: University of Chicago Press.